Fitting Learning Curves with Orthogonal Polynomials

Mark A. Sabol

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Instructional Technology Systems Technical Area
Training Research Laboratory

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U. S. Army

Research Institute for the Behavioral and Social Sciences

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

A FORTRAN program that generates orthogonal polynomials is listed. These polynomials can be used to analyze learning curves into components that can then be tested separately for significance (planned comparisons). The program is appropriate to situations in which the data are obtained from equally spaced intervals, as is naturally the case when the independent variable is trials, and also to situations involving unequal intervals (e.g., when data are available from the first, second, fourth, and eighth trials ∖(Continued)

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20. (Continued)

only). A demonstration of the component analysis is provided for both the
equal interval and unequal interval situation. These analyses result in
best-fit curves that reflect only those components that have proved signifi-
cant. An appendix lists all orthogonal polynomials for situations with
equal intervals and up to 20 trials. This listing greatly extends those
currently available in the statistical literature, which provide all poly-
nomials only for situations with up to 7 trials.

Technical Report 726

Fitting Learning Curves with Orthogonal Polynomials

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December 1986

The Instructional Technology Systems Technical Area of the U.S. Army Research Institute for the Behavioral and Social Sciences (ARI) performs research on learning processes that often involves the analysis of learning curves describing the way in which performance changes over trials or practice. It is often desirable to have a precise way of describing such curves, so that performance by different groups or under different conditions can be compared.

This report provides the tools with which a researcher can analyze a learning curve into its statistically significant components. A FORTRAN program that generates the correct coefficients (weights) needed to perform such a trend analysis is listed, and the procedure for applying those weights is described.

For convenience, the weights that are likely to be most commonly used (those for situations involving equal steps in practice) are provided as an appendix. However, the most valuable aspect of this product applies to the more unusual situation of unequally spaced practice. Generation of correct weights in such situations by hand can be laborious; the listed computer program handles such situations with ease. The program thus eliminates what might otherwise be a major obstacle in the application of trend analysis by researchers.

EDGAR M. JOHNSON
Technical Director

FITTING LEARNING CURVES WITH ORTHOGONAL POLYNOMIALS

EXECUTIVE SUMMARY

Requirement:

　　To facilitate the analysis of learning curves by researchers in the area of training.

Procedure:

　　The method of curve fitting known as trend analysis is described, and its use of special weights, known as coefficients of orthogonal polynomials, is discussed. A FORTRAN program that generates the necessary weights is listed.

Findings:

　　Those weights that are likely to be of most common use to researchers (those for situations in which performance data are available after equally spaced periods of practice) are provided in an appendix. Of greater potential value, however, is the ability of the FORTRAN program to provide the correct weights even for situations of unequally spaced practice.

Utilization of Findings:

　　It is hoped that this paper will foster the use of the trend analysis method of curve fitting by researchers who might otherwise be discouraged by the difficulty involved in finding the correct weights.

FITTING LEARNING CURVES WITH ORTHOGONAL POLYNOMIALS

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FITTING LEARNING CURVES WITH ORTHOGONAL POLYNOMIALS

INTRODUCTION

In a typical training study, one or more groups of subjects are presented with the same learning condition repeatedly, and the goal is to describe the learning curve that shows how performance varies as a function of the number of repetitions. The simplest description involves answering the question: is there a significant difference somewhere among the performances obtained on different trials? That is, is the curve significantly different from a horizontal line? This question would typically be answered with an overall analysis of variance, and the answer would be in the form of whether or not there was a significant effect of trials.

A more precise description is clearly to be preferred; learning curves generated by different subjects or groups of subjects or under the influence of various conditions could then be compared. One way of describing such a curve is through a technique of within-subjects (or repeated-measures) analysis of variance in which the variance due to the independent variable of repetitions or trials is partitioned into the variances due to several independent components (planned comparisons). For every degree of freedom in the original trials variable, one such comparison can be performed. The technique requires what are known as orthogonal polynomials.

A polynomial used for planned comparisons must be a string of positive and negative numbers (or zeros), one number for each level of the trials variable, whose sum is zero; such a string considered as a set of weights to be used in comparing sample means is often referred to as a contrast. For example, if there are three trials, the string -1, 0, 1 is a valid contrast. The individual numbers are referred to as the coefficients of the polynomial or contrast.

Orthogonality involves the relationship between two or more polynomials; to show that two polynomials are orthogonal, the coefficients of one polynomial are multiplied by the corresponding coefficients of the other, and their products are summed. If this sum is zero, the two polynomials are called orthogonal. Thus, the polynomial -1, 2, -1 is orthogonal to -1, 0, 1, since $(-1)(-1) + (2)(0) + (-1)(1) = 1 + 0 + -1 = 0$.

The term "orthogonal" arises because, with a three-coefficient polynomial, the coefficients can be taken as Cartesian coordinates for the end of a vector with its origin at 0,0,0; two orthogonal polynomials designate perpendicular vectors. Thus, the vectors -1, 2, -1 and -1, 0, 1 are perpendicular to each other and to the vector 1, 1, 1.

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The importance of orthogonality resides in its relationship to statistical independence of the sample comparisons. Hays (1973) discusses the issue as follows:

> Two comparisons satisfying this condition [the products of the weights sum to zero] are said to be orthogonal comparisons; orthogonality of two comparisons is equivalent to the statistical independence of sample comparisons only when the populations are normal, however. When two comparisons are statistically independent, the information each provides is actually nonredundant and unrelated to the information provided by the other... Thus, seeing if comparisons are orthogonal lets the experimenter judge whether or not he is gaining unrelated, nonoverlapping pieces of information about his experiment. (p. 588)

Assuming normal populations, the problem becomes that of generating the orthogonal polynomials. If there are n levels of the trials variable, it is possible to generate $n-1$ polynomials orthogonal to each other. The discussion, again, is from Hays (1973):

> It stands to reason that given any finite amount of data, only a finite set of questions may be asked of those data if one is to get nonredundant, non-overlapping answers. There is just so much information in any given set of data; once this information has been gained, asking further questions leads to answers that depend upon the answers already learned.
> This idea of the amount of information in a set of data has a statistical parallel in the number of possible independent (orthogonal) comparisons to be made among J means:
> Given J independent sample means, there can be no more than $J-1$ comparisons, each comparison being independent both of the grand mean and of each of the others. (p. 595)

The entire curve-fitting procedure involves obtaining all $n-1$ orthogonal polynomials, using them to test for the significance of each orthogonal component in the actual curve, and then producing an approximation to the actual curve that contains only those components that proved significant. The latter two steps are straightforward and are described below.

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In practice, the difficulty lies in obtaining the entire set
of orthogonal polynomials when the number of trials is at
all large.   Standard psychological statistics textbooks
(Kirk, 1968; Myers, 1969) list sets of n-1 orthogonal
polynomials only when n is less than or equal to five; Hays
(1973) extends this to seven.   Even the Biometrika Tables
for Statisticians, by Pearson and Hartley (1956), although
listing orthogonal polynomials for all cases out to n=52,
lists only the first six in each case, so that n-1
polynomials are listed only up to n=7.

This paper includes, in Appendix B, a listing of all
n-1 orthogonal polynomials for all cases out to n=20.   These
polynomials are correct when the data in the obtained curve
are from equally spaced intervals, as is naturally the case
when the independent variable is trials.   Sometimes,
however, analysis is desired of data from unequally spaced
intervals; for example, performance measures may be
available from the first, second, fourth, eighth, and
sixteenth trials.   For these situations, Appendix A includes
the listing of a computer program (based on an algorithm in
Acton, 1970) that generates orthogonal polynomials for any
interval spacing.   (Acton's discussion of his algorithm is
included here as Appendix C.)

PROCEDURE

The remainder of this paper is a demonstration of the
applied procedure (derived from Myers, 1969).   Calculations
will first be performed on a set of data obtained with equal
intervals and then on the same set of data, but with the
assumption of unequal intervals.

Equal Intervals

Assume that the data in Table 1 are derived from a
training experiment in which four subjects are measured on
five consecutive trials.   The calculations below the table
show the usual procedure for a within-subjects analysis of
variance (summarized in Table 2).

Table 1

Hypothetical Data from a Training Experiment

	Trials					
	1	2	3	4	5	
Subjects	x_1	x_2	x_3	x_4	x_5	
S_1	2	2	4	6	6	
S_2	2	3	6	9	4	
S_3	4	4	8	6	2	
S_4	2	4	8	8	4	
	—	—	—	—	—	overall
sum x	10	13	26	29	16	94
sum x^2	28	45	180	217	72	542
mean x	2.50	3.25	6.50	7.25	4.00	23.5
$\frac{(\text{sum } x)^2}{n}$	25.00	42.25	169.00	210.25	64.00	510.5
sum dev^2	3.00	2.75	11.00	6.75	8.00	31.50

$$SS_{total} = 542 - \frac{(94)^2}{20} = 542 - 441.8 = 100.2 \qquad SS_{within} = 31.50$$

$$SS_{trials} = SS_{total} - SS_{within} = 100.2 - 31.50 = 68.7$$

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Table 2

Summary Table for Within-Subjects Analysis of Variance

Source	SS	df	MS	F	p
Trials	68.7	4	17.18	8.18	<.005
Within	31.50	15	2.10		
Total	100.2	19			

We can now proceed to partition the four degrees of freedom for trials into the four independent components and test for the significance of each component. The coefficients of the four orthogonal polynomials with five equally-spaced levels of the independent variable (see Appendix B) are

Order						sum of squared coefficients
1)	-2	-1	0	1	2	10
2)	2	-1	-2	-1	2	14
3)	-1	2	0	-2	1	10
4)	1	-4	6	-4	1	70 .

These four polynomials are referred to as the linear, quadratic, cubic, and quartic components, respectively, or as the first second, third, and fourth order polynomials.

The formula for single degree of freedom sums of squares is

$$SS_{1df} = n(\underline{\text{sum of } c\bar{x})^2} \,, \quad \text{where } \underline{n} \text{ is the number of subjects, } \underline{c} \text{ is}$$
$$(\text{sum of } c^2) \qquad \text{a coefficient, and } \bar{x} \text{ is the trial mean.}$$

(If there are different numbers of subjects for different trials, the corresponding formula is

$$SS_{1df} = \frac{(\text{sum of } c\bar{x})^2}{\text{sum of } [c^2/n]} \,. \,)$$

Therefore,

$SS_{linear} =$

$$\frac{4[(-2)(2.50) + (-1)(3.25) + (0)(6.50) + (1)(7.25) + (2)(4.00)]^2}{10}$$
$$= 4[7]^2/10 = 19.6$$

$SS_{quadratic} =$

$$\frac{4[(2)(2.50) + (-1)(3.25) + (-2)(6.50) + (-1)(7.25) + (2)(4.00)]^2}{14}$$
$$= 4[-10.5]^2/14 = 31.5$$

$SS_{cubic} =$

$$\frac{4[(-1)(2.50) + (2)(3.25) + (0)(6.50) + (-2)(7.25) + (1)(4.00)]^2}{10}$$
$$= 4[-6.5]^2/10 = 16.9$$

$SS_{quartic} =$

$$\frac{4[(1)(2.50) + (-4)(3.25) + (6)(6.50) + (-4)(7.25) + (1)(4.00)]^2}{70}$$
$$= 4[3.5]^2/70 = 0.7 \,.$$

Note that the total of the four component sums of squares is equal to the sums of squares calculated above for the trials variable: 19.6 + 31.5 + 16.9 + 0.7 = 68.7. This must be so if the coefficients of the orthogonal polynomials have been determined correctly.

These component SSs become F tests using the "mean square within" term from the overall analysis of variance, as follows:

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$$\underline{MS}\text{linear} \quad = \frac{19.6}{1} = 19.6 \quad \underline{F}(1,15) = \frac{19.6}{2.1} = 9.33 \quad (\underline{p}<.01)$$

$$\underline{MS}\text{quadratic} = \frac{31.5}{1} = 31.5 \quad \underline{F}(1,15) = \frac{31.5}{2.1} = 15.00 \quad (\underline{p}<.01)$$

$$\underline{MS}\text{cubic} \quad = \frac{16.9}{1} = 16.9 \quad \underline{F}(1,15) = \frac{16.9}{2.1} = 8.05 \quad (\underline{p}<.05)$$

$$\underline{MS}\text{quartic} \quad = \frac{0.7}{1} = 0.7 \quad \underline{F}(1,15) = \frac{0.7}{2.1} = 0.33 \quad (\text{ n.s.}) .$$

The final step is to generate the points on a curve based on these components. The curve that would best fit the original data would be one which contained all four components; in this case, however, the quartic component will not be included, since it failed to reach significance. For each of the other components, a weight is first calculated from the formula

$$\underline{b} = \frac{\text{sum of } c\bar{x}}{\text{sum of } c^2} .$$

These are

$$\underline{b}\text{linear} = 7/10 = 0.70$$

$$\underline{b}\text{quadratic} = -10.5/14 = -0.75$$

$$\underline{b}\text{cubic} = -6.5/10 = -.65 .$$

These weights are combined with the overall mean (GM) and the polynomial coefficients to yield the points on the approximate curve:

$$Y_1 = GM + (\underline{b}\text{lin})(c_1, \text{lin}) + (\underline{b}\text{quad})(c_1, \text{quad}) + (\underline{b}\text{cub})(c_1, \text{cubic})$$

$$Y_2 = GM + (\underline{b}\text{lin})(c_2, \text{lin}) + (\underline{b}\text{quad})(c_2, \text{quad}) + (\underline{b}\text{cub})(c_2, \text{cubic})$$

$$Y_3 = GM + (\underline{b}\text{lin})(c_3, \text{lin}) + (\underline{b}\text{quad})(c_3, \text{quad}) + (\underline{b}\text{cub})(c_3, \text{cubic})$$

$$Y_4 = GM + (\underline{b}\text{lin})(c_4, \text{lin}) + (\underline{b}\text{quad})(c_4, \text{quad}) + (\underline{b}\text{cub})(c_4, \text{cubic})$$

$$Y_5 = GM + (\underline{b}\text{lin})(c_5, \text{lin}) + (\underline{b}\text{quad})(c_5, \text{quad}) + (\underline{b}\text{cub})(c_5, \text{cubic}),$$

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where $c_{1,lin}$ is the first coefficient of the linear polynomial, $c_{2,lin}$ is the second coefficient of the linear polynomial, etc.

In the present case, we have

$$Y_1 = 4.70 + (.70)(-2) + (-.75)(2) + (-.65)(-1) = 2.45$$

$$Y_2 = 4.70 + (.70)(-1) + (-.75)(-1) + (-.65)(2) = 3.45$$

$$Y_3 = 4.70 + (.70)(0) + (-.75)(-2) + (-.65)(0) = 6.20$$

$$Y_4 = 4.70 + (.70)(1) + (-.75)(-1) + (-.65)(-2) = 7.45$$

$$Y_5 = 4.70 + (.70)(2) + (-.75)(2) + (-.65)(1) = 3.95 .$$

A comparison of these points generated for the approximate curve with the original means (2.50, 3.25, 6.50, 7.25, and 4.00; see Figure 1) reveals the goodness of fit.
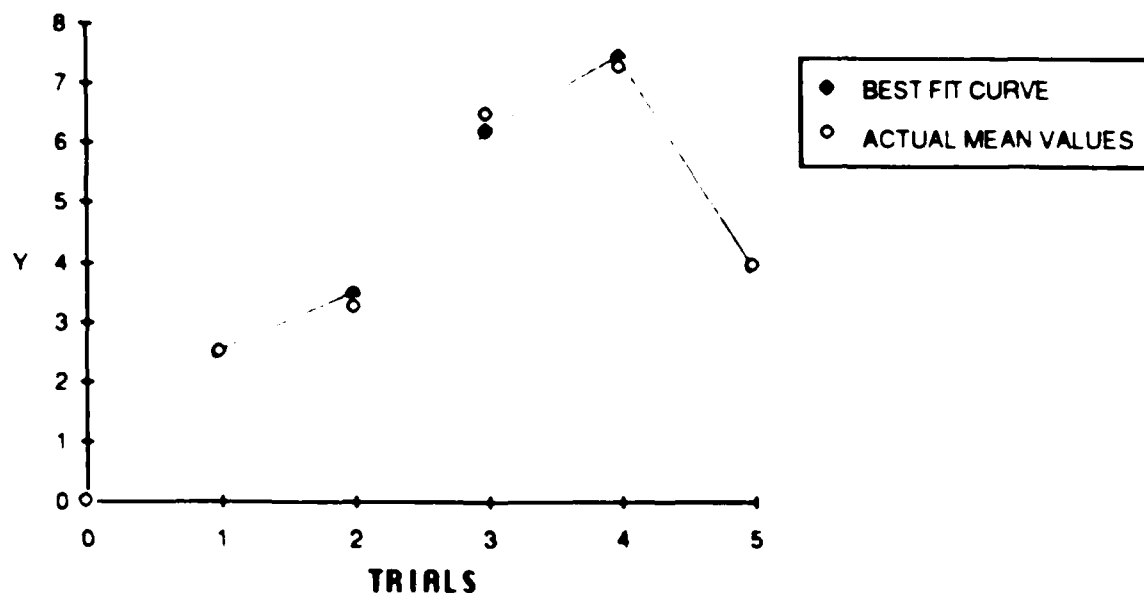
Figure 1. Hypothetical data, equal intervals in the independent variable.

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Unequal Intervals

The same calculations will now be shown for the situation in which the interval values are not equal; that is, the levels of the trials variable are not equally spaced. Specifically, we will assume that the data shown in Table 1 above were derived from trials 1, 2, 4, 8, and 16.

The FORTRAN program in Appendix A was run to generate the appropriate orthogonal polynomials. The coefficients are:

Order						sum of squared coefficients
1)	-26	-21	-11	9	49	3,720
2)	30	11	-19	-47	25	4,216
3)	-176	76	252	-181	29	133,858
4)	64	-120	70	-15	1	23,622 .

Applying the same formula used above for the sums of squares for the single degree of freedom components, we obtain:

$$SS_{1df} = \frac{n(\text{sum of } c\bar{x})^2}{\text{sum of } c^2} =$$

$SS_{linear} =$

$$\frac{4[(-26)(2.50)+(-21)(3.25)+(-11)(6.50)+(9)(7.25)+(49)(4.00)]^2}{3720}$$

$$= \frac{4[56.5]^2}{3720} = 3.43$$

$SS_{quadratic} =$

$$\frac{4[(30)(2.50)+(11)(3.25)+(-19)(6.50)+(-47)(7.25)+(25)(4.00)]^2}{4216}$$

$$= \frac{4[-253.5]^2}{4216} = 60.97$$

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SS_{cubic} =

$$\frac{4[(-176)(2.50)+(\ \ 76)(3.25)+(\ \ 252)(6.50)+(-181)(7.25)+(\ \ 29)(4.00)]^2}{133858}$$

$$= \frac{4[248.75]^2}{133858} = 1.85$$

$SS_{quartic}$ =

$$\frac{4[(\ \ 64)(2.50)+(-120)(3.25)+(\ \ 70)(6.50)+(\ -15)(7.25)+(\ \ 1)(4.00)]^2}{23622}$$

$$= \frac{4[120.25]^2}{23622} = 2.49 \ .$$

(Note again that these sum to the sums of squares for trials: 3.43 + 60.97 + 1.85 + 2.49 = 68.74 .)

Dividing by the overall mean square within, we obtain

$$\underline{F}_{(1,15)linear} = \frac{3.42}{2.1} = 1.63 \qquad (\text{n.s.})$$

$$\underline{F}_{(1,15)quadratic} = \frac{60.97}{2.1} = 29.03 \qquad (\underline{p}<.001)$$

$$\underline{F}_{(1,15)cubic} = \frac{1.85}{2.1} = 0.88 \qquad (\text{n.s.})$$

$$\underline{F}_{(1,15)quartic} = \frac{2.49}{2.1} = 1.19 \qquad (\text{n.s.}).$$

Since $\underline{b}_{quadratic} = \dfrac{\text{sum of } c\bar{x}}{\text{sum of } c^2} = -\dfrac{253.5}{4216} = -.0601,$

$Y_1 = 4.70 + (-.0601)(\ 30) = 4.70 - 1.803 = 2.90$

$Y_2 = 4.70 + (-.0601)(\ 11) = 4.70 - 0.661 = 4.04$

$Y_3 = 4.70 + (-.0601)(-19) = 4.70 + 1.142 = 5.84$

$Y_4 = 4.70 + (-.0601)(-47) = 4.70 + 2.825 = 7.52$

$Y_5 = 4.70 + (-.0601)(\ 25) = 4.70 - 1.502 = 3.20 \ ,$

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which are to be compared with the original trial means (2.50, 3.25, 6.50, 7.25, and 4.00; see Figure 2) to evaluate the goodness of fit.
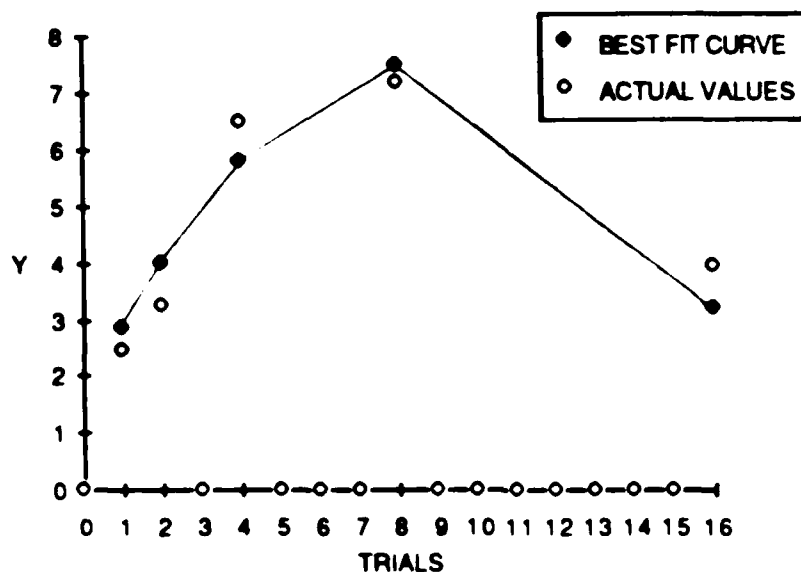
Figure 2. Hypothetical data, unequal intervals in the independent variable.

SUMMARY

This paper has provided a demonstration of a procedure for curve-fitting (trend analysis) based upon planned comparisons. While this procedure is applicable to data obtained from any situation in which there is a multi-leveled quantitative independent variable, the ideal situation for its application is that of a learning experiment, in which there is a single independent variable (trials) with a large number of levels.

The advantage of the procedure is its ability to separately test the significance of the different trends which combine to form the overall curve shape. Those trends that prove significant are then recombined to produce a new curve. This curve is, in general, not the curve that would provide the best fit to the data points; a perfect fit would be obtained by always including all trends, whether significant or not, in the reconstructed curve. However,

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the function that results from the recombination of significant trends provides an estimate of population values; the variability of actual data points around the values predicted by that function is assumed to be error variance. Comparison "by eye" of one group's learning curve with that of another group would therefore be more valid if the reconstructed curves were compared, rather than the curves composed of the original observed data points.

Precise statistical comparison of group curves (beyond a mere listing of which trends were significant for the different groups) requires the test for interaction within a mixed design analysis of variance; such tests involve the same orthogonal polynomials generated here, but their application is complex to a point beyond the scope of this paper. The interested reader is referred to the excellent discussion by Myers (1969).

The contribution of this paper is not the description of application procedures, but rather the provision of the coefficients of orthogonal polynomials in the appendix. These extend far beyond those currently available in the literature. Of potentially even greater value to the researcher is the listing of a FORTRAN program that generates such coefficients for those situations in which the levels of the independent variable are not equally spaced. It is hoped that this paper will foster the use of the trend analysis method of curve fitting by researchers who would otherwise be discouraged by the difficulty involved in finding or generating the correct coefficients.

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FORTRAN PROGRAM TO GENERATE ORTHOGONAL POLYNOMIALS
WITH EQUAL AND UNEQUAL INTERVALS

```
      real*16 p(25,26),g(25),nx(25),a(25),b(25),dmo,unit,newtemp(25),
     1tabs,tmin,nevp(25),th(25),totsq
      dimension ntemp(50)
      open(unit=3,name='newact.lis',type='new',form='formatted')
c
c SET NUMBER OF INTERVAL VALUES HERE. MAKE niv = NUMBER - 1.
c TO PRODUCE ALL POLYNOMIALS, SET NIV = 1 AND USE LAST LINE BEFORE "END"
c
      niv = 24
4     niv = niv + 1
c
c GENERATING POLYNOMIAL COEFFICIENTS:
c
c START BY DEFINING INTERVAL VALUES --
c
c FOR EQUAL INTERVALS BEGINNING WITH X=1 (X=1,2,3,....),
c  USING THE FOLLOWING:
      do i=1,niv
        nx(i) = i
      end do
c
c FOR OTHER INTERVAL VALUES, DEFINE THEM EXPLICITLY
c  AS BELOW (e.g., X=1,2,4,8,16):
c      nx(1)=1
c      nx(2)=2
c      nx(3)=4
c      nx(4)=8
c      nx(5)=16
c
      do i=1,niv
        p(1,i) = 1
      end do
      do j=1,niv
        g(j) = 0
        do i=1,niv
          g(j) = g(j) + p(j,i)**2
        end do
        a(j) = 0
        do i=1,niv
          a(j) = a(j) + nx(i)*(p(j,i)**2)/g(j)
        end do
        if(j.gt.1)goto 1
        b(1) = 0
        goto 2
```

```fortran
1         b(j) = g(j)/g(j-1)
2         do i=1,niv
              p(j+1,i) = (nx(i) - a(j))*p(j,i) - b(j)*p(j-1,i)
          end do
          if((j-1).eq.0)goto 99
          write(3,123)j-1
123       format(/'_____[]_____
     1_____',/' Order = ',i2)
c
c COEFFICIENTS HAVE BEEN GENERATED; NOW THEY ARE CONVERTED TO INTEGERS
c
          unit = 1.00000000000000000000
          tmin = 99999999999999999999999.99
          do i=1,niv
              if(abs(p(j,i)).lt.(0.001))goto 3
              th(i) = abs(p(j,i))
              if(th(i).lt.tmin)tmin = th(i)
3         end do
          do i=1,niv
              newp(i) = p(j,i)/tmin
          end do
89        do newi=1,1000*niv
              do ic=1,niv
                  newtemp(ic) = newp(ic)*newi
              end do
              do ic=1,niv
                  dmo = mod(newtemp(ic),unit)
                  if(abs(dmo).gt.(0.01).and.abs(dmo).lt.(0.99))goto 93
              end do
            goto 95
93          continue
          end do
c
c COEFFICIENTS PRINTED HERE IF INTEGER VALUES COULD NOT BE FOUND
c
          write(3,124)(newp(i),i=1,niv)
124       format(/4(1x,f19.3))
          goto 82
95        totsq = 0
          do i=1,niv
              ntemp(i) = nint(newtemp(i))
              totsq = totsq + newtemp(i)**2
          end do
c
c COEFFIENTS PRINTED HERE IF INTEGER CONVERSION WAS SUCCESSFUL
c
          write(3,126)(ntemp(ic),ic=1,niv)
126       format(/4(1x,i19))
          write(3,133) totsq
133       format(/' sum of coefficients squared = ',f26.2)
82        continue
99    end do
      write(3,125)
125   format(/'_____[]_____
     1_____',')
```

```
c
c   USE THE FOLLOWING COMMAND ONLY IF YOU ARE GENERATING
c   ALL POLYNOMIALS FROM niv = 2 TO niv = 25
c     NOTE: GENERATION TIMES BECOME VERY LARGE WHEN niv > 20
c           DUE TO DIFFICULTY IN CONVERSION TO INTEGERS;
c           BUT INTEGER CONVERSION IS NOT NECESSARY AND CAN BE DELETED
c
c     if(niv.lt.25)goto 4
      end
```

COEFFICIENTS OF ORTHOGONAL POLYNOMIALS
n = 2 TO 20, ORDERS 1 TO n-1

n = 2

Order = 1

-1	1

sum of coefficients squared =   2.00

n = 3

Order = 1

-1	0	1

sum of coefficients squared =   2.00

Order = 2

1	-2	1

sum of coefficients squared =   6.00

n = 4

Order = 1

-3	-1	1	3

sum of coefficients squared =   20.00

Order = 2

1	-1	-1	1

sum of coefficients squared =   4.00

Order = 3

-1	3	-3	1

sum of coefficients squared =   20.00

n = 5

Order = 1

-2	-1	0	1
2			

sum of coefficients squared =   10.00

Order = 2

2	-1	-2	-1
2			

sum of coefficients squared =   14.00

Order = 3

-1	2	0	-2
1			

sum of coefficients squared =   10.00

Order = 4

	1	-4	6	-4
	1			

sum of coefficients squared =   70.00

---

n = 6

Order = 1

	-5	-3	-1	1
	3	5		

sum of coefficients squared =   70.00

Order = 2

	5	-1	-4	-4
	-1	5		

sum of coefficients squared =   84.00

Order = 3

	-5	7	4	-4
	-7	5		

sum of coefficients squared =   180.00

Order = 4

	1	-3	2	2
	-3	1		

sum of coefficients squared =   28.00

Order = 5

	-1	5	-10	10
	-5	1		

sum of coefficients squared =   252.00

---

n = 7

Order = 1

	-3	-2	-1	0
	1	2	3	

sum of coefficients squared =   28.00

Order = 2

	5	0	-3	-4
	-3	0	5	

sum of coefficients squared =   84.00

Order = 3

	-1	1	1	0
	-1	-1	1	

sum of coefficients squared =   6.00

Order = 4

	3	-7	1	6
	1	-7	3	

sum of coefficients squared =   154.00

B-2

Order = 5			
-1	4	-5	0
5	-4	1	
sum of coefficients squared =		84.00	

Order = 6			
1	-6	15	-20
15	-6	1	
sum of coefficients squared =		924.00	

n = 8

Order = 1			
-7	-5	-3	-1
1	3	5	7
sum of coefficients squared =		168.00	

Order = 2			
7	1	-3	-5
-5	-3	1	7
sum of coefficients squared =		168.00	

Order = 3			
-7	5	7	3
-3	-7	-5	7
sum of coefficients squared =		264.00	

Order = 4			
7	-13	-3	9
9	-3	-13	7
sum of coefficients squared =		616.00	

Order = 5			
-7	23	-17	-15
15	17	-23	7
sum of coefficients squared =		2184.00	

Order = 6			
1	-5	9	-5
-5	9	-5	1
sum of coefficients squared =		264.00	

Order = 7			
-1	7	-21	35
-35	21	-7	1
sum of coefficients squared =		3432.00	

n = 9

Order = 1			
-4	-3	-2	-1
0	1	2	3
4			
sum of coefficients squared =		60.00	

B-3

```
Order =  2
              28              7              -8             -17
             -20            -17              -8               7
              28
        sum of coefficients squared =      2772.00
```

```
Order =  3
             -14              7              13               9
               0             -9             -13              -7
              14
        sum of coefficients squared =       990.00
```

```
Order =  4
              14            -21             -11               9
              18              9             -11             -21
              14
        sum of coefficients squared =      2002.00
```

```
Order =  5
              -4             11              -4              -9
               0              9               4             -11
               4
        sum of coefficients squared =       468.00
```

```
Order =  6
               4            -17              22               1
             -20              1              22             -17
               4
        sum of coefficients squared =      1980.00
```

```
Order =  7
              -1              6             -14              14
               0            -14              14              -6
               1
        sum of coefficients squared =       858.00
```

```
Order =  8
               1             -8              28             -56
              70            -56              28              -8
               1
        sum of coefficients squared =     12870.00
```

```
n = 10
```

```
Order =  1
              -9             -7              -5              -3
              -1              1               3               5
               7              9
        sum of coefficients squared =       330.00
```

```
Order =  2
               6              2              -1              -3
              -4             -4              -3              -1
               2              6
```

sum of coefficients squared =		132.00	

Order = 3

-42	14	35	31
12	-12	-31	-35
-14	42		
sum of coefficients squared =		8580.00	

Order = 4

18	-22	-17	3
18	18	3	-17
-22	18		
sum of coefficients squared =		2860.00	

Order = 5

-6	14	-1	-11
-6	6	11	1
-14	6		
sum of coefficients squared =		780.00	

Order = 6

3	-11	10	6
-8	-8	6	10
-11	3		
sum of coefficients squared =		660.00	

Order = 7

-9	47	-86	42
56	-56	-42	86
-47	9		
sum of coefficients squared =		29172.00	

Order = 8

1	-7	20	-28
14	14	-28	20
-7	1		
sum of coefficients squared =		2860.00	

Order = 9

-1	9	-36	84
-126	126	-84	36
-9	1		
sum of coefficients squared =		48620.00	

---

n = 11

Order = 1

-5	-4	-3	-2
-1	0	1	2
3	4	5	
sum of coefficients squared =		110.00	

Order = 2

15	6	-1	-6
-9	-10	-9	-6

	-1	6	15	
	sum of coefficients squared =		858.00	

Order = 3

	-30	6	22	23
	14	0	-14	-23
	-22	-6	30	
	sum of coefficients squared =		4290.00	

Order = 4

	6	-6	-6	-1
	4	6	4	-1
	-6	-6	6	
	sum of coefficients squared =		286.00	

Order = 5

	-3	6	1	-4
	-4	0	4	4
	-1	-6	3	
	sum of coefficients squared =		156.00	

Order = 6

	15	-48	29	36
	-12	-40	-12	36
	29	-48	15	
	sum of coefficients squared =		11220.00	

Order = 7

	-5	23	-33	2
	28	0	-28	-2
	33	-23	5	
	sum of coefficients squared =		4862.00	

Order = 8

	5	-31	73	-68
	-14	70	-14	-68
	73	-31	5	
	sum of coefficients squared =		27170.00	

Order = 9

	-1	8	-27	48
	-42	0	42	-48
	27	-8	1	
	sum of coefficients squared =		9724.00	

Order = 10

	1	-10	45	-120
	210	-252	210	-120
	45	-10	1	
	sum of coefficients squared =		184756.00	

---

n = 12

Order = 1

	-11	-9	-7	-5

-3	-1	1	3
5	7	9	11

sum of coefficients squared = 572.00

Order = 2

55	25	1	-17
-29	-35	-35	-29
-17	1	25	55

sum of coefficients squared = 12012.00

Order = 3

-33	3	21	25
19	7	-7	-9
-25	-21	-3	33

sum of coefficients squared = 5148.00

Order = 4

33	-27	-33	-13
12	28	28	12
-13	-33	-27	33

sum of coefficients squared = 8008.00

Order = 5

-33	57	21	-29
-44	-20	20	44
29	-21	-57	33

sum of coefficients squared = 15912.00

Order = 6

11	-31	11	25
4	-20	-20	4
25	11	-31	11

sum of coefficients squared = 4488.00

Order = 7

-55	225	-251	-83
204	140	-140	-204
83	251	-225	55

sum of coefficients squared = 369512.00

Order = 8

11	-61	119	-65
-74	70	70	-74
-65	119	-61	11

sum of coefficients squared = 65208.00

Order = 9

-11	79	-227	303
-102	-210	210	102
-303	227	-79	11

sum of coefficients squared = 408408.00

Order = 10

1	-9	35	-75
90	-42	-42	90

-75	35	-9	1

sum of coefficients squared = 33592.00

Order = 11

-1	11	-55	165
-330	462	-462	330
-165	55	-11	1

sum of coefficients squared = 705432.00

---

n = 13

Order = 1

-6	-5	-4	-3
-2	-1	0	1
2	3	4	5
6			

sum of coefficients squared = 182.00

Order = 2

22	11	2	-5
-10	-13	-14	-13
-10	-5	2	11
22			

sum of coefficients squared = 2002.00

Order = 3

-11	0	6	8
7	4	0	-4
-7	-8	-6	0
11			

sum of coefficients squared = 572.00

Order = 4

99	-66	-96	-54
11	64	84	64
11	-54	-96	-66
99			

sum of coefficients squared = 68068.00

Order = 5

-22	33	18	-11
-26	-20	0	20
26	11	-18	-33
22			

sum of coefficients squared = 6188.00

Order = 6

22	-55	8	43
22	-20	-40	-20
22	43	8	-55
22			

sum of coefficients squared = 14212.00

Order = 7

-33	121	-103	-75

65	100	0	-100
-65	75	103	-121
33			

sum of coefficients squared = 92378.00

Order = 8

11	-55	89	-19
-71	10	70	10
-71	-19	89	-55
11			

sum of coefficients squared = 38038.00

Order = 9

-2	13	-32	31
6	-30	0	30
-6	-31	32	-13
2			

sum of coefficients squared = 6188.00

Order = 10

6	-49	166	-285
210	78	-252	78
210	-285	166	-49
6			

sum of coefficients squared = 386308.00

Order = 11

-1	10	-44	110
-165	132	0	-132
165	-110	44	-10
1			

sum of coefficients squared = 117572.00

Order = 12

1	-12	66	-220
495	-792	924	-792
495	-220	66	-12
1			

sum of coefficients squared = 2704156.00

---

n = 14

Order = 1

-13	-11	-9	-7
-5	-3	-1	1
3	5	7	9
11	13		

sum of coefficients squared = 910.00

Order = 2

13	7	2	-2
-5	-7	-8	-8
-7	-5	-2	2
7	13		

sum of coefficients squared = 728.00

Order = 3

-143	-11	66	98
95	67	24	-24
-67	-95	-98	-66
11	143		

sum of coefficients squared =   97240.00

Order = 4

143	-77	-132	-92
-13	63	108	108
63	-13	-92	-132
-77	143		

sum of coefficients squared =   136136.00

Order = 5

-143	187	132	-28
-139	-145	-60	60
145	139	28	-132
-187	143		

sum of coefficients squared =   235144.00

Order = 6

143	-319	-11	227
185	-25	-200	-200
-25	185	227	-11
-319	143		

sum of coefficients squared =   497420.00

Order = 7

-143	473	-297	-353
95	375	200	-200
-375	-95	353	297
-473	143		

sum of coefficients squared =   1293292.00

Order = 8

13	-59	79	7
-65	-25	50	50
-25	-65	7	79
-59	13		

sum of coefficients squared =   34580.00

Order = 9

-13	77	-163	107
89	-105	-90	90
105	-89	-107	163
-77	13		

sum of coefficients squared =   142324.00

Order = 10

13	-97	288	-392
125	279	-216	-216
279	125	-392	288
-97	13		

sum of coefficients squared =   772616.00

Order = 11

-13	119	-464	968
-1045	231	792	-792
-231	1045	-968	464
-119	13		

sum of coefficients squared =   5878600.00

Order = 12

1	-11	54	-154
275	-297	132	132
-297	275	-154	54
-11	1		

sum of coefficients squared =   416024.00

Order = 13

-1	13	-78	286
-715	1287	-1716	1716
-1287	715	-286	78
-13	1		

sum of coefficients squared =   10400600.00

---

n = 15

Order = 1

-7	-6	-5	-4
-3	-2	-1	0
1	2	3	4
5	6	7	

sum of coefficients squared =   280.00

Order = 2

91	52	19	-8
-29	-44	-53	-56
-53	-44	-29	-8
19	52	91	

sum of coefficients squared =   37128.00

Order = 3

-91	-13	35	58
61	49	27	0
-27	-49	-61	-58
-35	13	91	

sum of coefficients squared =   39780.00

Order = 4

1001	-429	-869	-704
-249	251	621	756
621	251	-249	-704
-869	-429	1001	

sum of coefficients squared =   6466460.00

Order = 5

-1001	1144	979	44

-751	-1000	-675	0
675	1000	751	-44
-979	-1144	1001	

sum of coefficients squared = 10581480.00

Order = 6

143	-286	-55	176
197	50	-125	-200
-125	50	197	176
-55	-286	143	

sum of coefficients squared = 426360.00

Order = 7

-13	39	-17	-31
-3	25	25	0
-25	-25	3	31
17	-39	13	

sum of coefficients squared = 8398.00

Order = 8

91	-377	415	157
-311	-275	125	350
125	-275	-311	157
415	-377	91	

sum of coefficients squared = 1193010.00

Order = 9

-91	494	-901	344
659	-250	-675	0
675	250	-659	-344
901	-494	91	

sum of coefficients squared = 4269720.00

Order = 10

91	-624	1631	-1724
-159	1568	-27	-1512
-27	1568	-159	-1724
1631	-624	91	

sum of coefficients squared = 19315400.00

Order = 11

-7	59	-205	356
-253	-121	297	0
-297	121	253	-356
205	-59	7	

sum of coefficients squared = 678300.00

Order = 12

7	-71	313	-766
1067	-649	-363	924
-363	-649	1067	-766
313	-71	7	

sum of coefficients squared = 5616324.00

Order = 13

-1	12	-65	208
-429	572	-429	0
429	-572	429	-208
65	-12	1	
sum of coefficients squared =		1485800.00	

Order = 14

1	-14	91	-364
1001	-2002	3003	-3432
3003	-2002	1001	-364
91	-14	1	
sum of coefficients squared =		40116600.00	

n = 16

Order = 1

-15	-13	-11	-9
-7	-5	-3	-1
1	3	5	7
9	11	13	15
sum of coefficients squared =		1360.00	

Order = 2

35	21	9	-1
-9	-15	-19	-21
-21	-19	-15	-9
-1	9	21	35
sum of coefficients squared =		5712.00	

Order = 3

-455	-91	143	267
301	265	179	63
-63	-179	-265	-301
-267	-143	91	455
sum of coefficients squared =		1007760.00	

Order = 4

273	-91	-221	-201
-101	23	129	189
189	129	23	-101
-201	-221	-91	273
sum of coefficients squared =		470288.00	

Order = 5

-143	143	143	33
-77	-131	-115	-45
45	115	131	77
-33	-143	-143	143
sum of coefficients squared =		201552.00	

Order = 6

65	-117	-39	59
87	45	-25	-75
-75	-25	45	87
59	-39	-117	65

sum of coefficients squared = 77520.00

Order = 7

-195	533	-143	-423
-157	235	375	175
-175	-375	-235	157
423	143	-533	195

sum of coefficients squared = 1545232.00

Order = 8

65	-247	221	149
-133	-205	-25	175
175	-25	-205	-133
149	221	-247	65

sum of coefficients squared = · 454480.00

Order = 9

-91	455	-715	95
575	53	-505	-315
315	505	-53	-575
-95	715	-455	91

sum of coefficients squared = 2846480.00

Order = 10

21	-133	307	-243
-133	229	141	-189
-189	141	229	-133
-243	307	-133	21

sum of coefficients squared = 594320.00

Order = 11

-35	273	-849	1219
-453	-825	649	693
-693	-649	825	453
-1219	849	-273	35

sum of coefficients squared = 8139600.00

Order = 12

5	-47	187	-393
413	-55	-341	231
231	-341	-55	413
-393	187	-47	5

sum of coefficients squared = 1069776.00

Order = 13

-15	167	-821	2301
-3887	3575	-429	-3003
3003	429	-3575	3887
-2301	821	-167	15

sum of coefficients squared = 86176400.00

Order = 14

1	-13	77	-273
637	-1001	1001	-429
-429	1001	-1001	637

-273	77	-13	1

sum of coefficients squared = 5348880.00

Order = 15

-1	15	-105	455
-1365	3003	-5005	6435
-6435	5005	-3003	1365
-455	105	-15	1

sum of coefficients squared = 155117520.00

---

n = 17

Order = 1

-8	-7	-6	-5
-4	-3	-2	-1
0	1	2	3
4	5	6	7
8			

sum of coefficients squared = 408.00

Order = 2

40	25	12	1
-8	-15	-20	-23
-24	-23	-20	-15
-8	1	12	25
40			

sum of coefficients squared = 7752.00

Order = 3

-28	-7	7	15
18	17	13	7
0	-7	-13	-17
-18	-15	-7	7
28			

sum of coefficients squared = 3876.00

Order = 4

52	-13	-39	-39
-24	-3	17	31
36	31	17	-3
-24	-39	-39	-13
52			

sum of coefficients squared = 16796.00

Order = 5

-104	91	104	39
-36	-83	-88	-55
0	55	88	83
36	-39	-104	-91
104			

sum of coefficients squared = 100776.00

Order = 6

104	-169	-78	65
128	93	2	-85

-120	-85	2	93
128	65	-78	-169
104			

sum of coefficients squared = 178296.00

Order = 7

-130	325	-39	-247
-149	75	215	175
0	-175	-215	-75
149	247	39	-325
130			

sum of coefficients squared = 579462.00

Order = 8

26	-91	65	65
-25	-73	-37	35
70	35	-37	-73
-25	65	65	-91
26			

sum of coefficients squared = 56810.00

Order = 9

-8	37	-50	-5
40	19	-26	-35
0	35	26	-19
-40	5	50	-37
8			

sum of coefficients squared = 15640.00

Order = 10

56	-329	672	-373
-428	309	464	-119
-504	-119	464	309
-428	-373	672	-329
56			

sum of coefficients squared = 2674440.00

Order = 11

-4	29	-81	95
-4	-81	11	77
0	-77	-11	81
4	-95	81	-29
4			

sum of coefficients squared = 58140.00

Order = 12

20	-175	631	-1137
846	365	-935	-77
924	-77	-935	365
846	-1137	631	-175
20			

sum of coefficients squared = 7755876.00

Order = 13

-8	83	-372	915

-1248	663	572	-1001
0	1001	-572	-663
1248	-915	372	-83
8			

sum of coefficients squared = 8617640.00

Order = 14

8	-97	526	-1659
3276	-3913	2002	1573
-3432	1573	2002	-3913
3276	-1659	526	-97
8			

sum of coefficients squared = 82907640.00

Order = 15

-1	14	-90	350
-910	1638	-2002	1430
0	-1430	2002	-1638
910	-350	90	-14
1			

sum of coefficients squared = 19389690.00

Order = 16

1	-16	120	-560
1820	-4368	8008	-11440
12870	-11440	8008	-4368
1820	-560	120	-16
1			

sum of coefficients squared = 601080390.00

---

n = 18

Order = 1

-17	-15	-13	-11
-9	-7	-5	-3
-1	1	3	5
7	9	11	13
15	17		

sum of coefficients squared = 1938.00

Order = 2

68	44	23	5
-10	-22	-31	-37
-40	-40	-37	-31
-22	-10	5	23
44	68		

sum of coefficients squared = 23256.00

Order = 3

-68	-20	13	33
42	42	35	23
8	-8	-23	-35
-42	-42	-33	-13
20	68		

sum of coefficients squared = 23256.00

Order = 4			
68	-12	-47	-51
-36	-12	13	33
44	44	33	13
-12	-36	-51	-47
-12	68		
sum of coefficients squared =		25424.00	

Order = 5			
-884	676	871	429
-156	-588	-733	-583
-220	220	583	733
588	156	-429	-871
-676	884		
sum of coefficients squared =		6953544.00	

Order = 6			
442	-650	-377	169
481	439	145	-209
-440	-440	-209	145
439	481	169	-377
-650	442		
sum of coefficients squared =		2941884.00	

Order = 7			
-442	1014	13	-715
-585	31	563	651
280	-280	-651	-563
-31	585	715	-13
-1014	442		
sum of coefficients squared =		5794620.00	

Order = 8			
34	-110	61	85
-5	-77	-65	7
70	70	7	-65
-77	-5	85	61
-110	34		
sum of coefficients squared =		78660.00	

Order = 9			
-34	146	-169	-55
125	107	-43	-133
-70	70	133	43
-107	-125	55	169
-146	34		
sum of coefficients squared =		211140.00	

Order = 10			
68	-372	673	-229
-504	112	497	147
-392	-392	147	497
112	-504	-229	673
-372	68		

sum of coefficients squared =   2674440.00

Order = 11

-68	460	-1157	1087
328	-992	-365	847
616	-616	-847	365
992	-328	-1087	1157
-460	68		

sum of coefficients squared =   10116360.00

Order = 12

68	-556	1823	-2805
1290	1722	-1619	-1463
1540	1540	-1463	-1619
1722	1290	-2805	1823
-556	68		

sum of coefficients squared =   46535256.00

Order = 13

-68	660	-2707	5847
-6282	858	5135	-3003
-4004	4004	3003	-5135
-858	6282	-5847	2707
-660	68		

sum of coefficients squared = 267146840.00

Order = 14

17	-193	962	-2706
4494	-3822	-182	3718
-2288	-2288	3718	-182
-3822	4494	-2706	962
-193	17		

sum of coefficients squared = 124361460.00

Order = 15

-17	223	-1322	4630
-10430	15106	-12194	286
11440	-11440	-286	12194
-15106	10430	-4630	1322
-223	17		

sum of coefficients squared =   1279719540.00

Order = 16

1	-15	104	-440
1260	-2548	3640	-3432
1430	1430	-3432	3640
-2548	1260	-440	104
-15	1		

sum of coefficients squared =   70715340.00

Order = 17

-1	17	-136	680
-2380	6188	-12376	19448
-24310	24310	-19448	12376
-6188	2380	-680	136

| | -17 | | 1 |
| sum of coefficients squared = | | 2333606220.00 | |

n = 19

Order = 1

-9	-8	-7	-6
-5	-4	-3	-2
-1	0	1	2
3	4	5	6
7	8	9	

sum of coefficients squared = 570.00

Order = 2

51	34	19	6
-5	-14	-21	-26
-29	-30	-29	-26
-21	-14	-5	6
19	34	51	

sum of coefficients squared = 13566.00

Order = 3

-204	-68	28	89
120	126	112	83
44	0	-44	-83
-112	-126	-120	-89
-28	68	204	

sum of coefficients squared = 213180.00

Order = 4

612	-68	-388	-453
-354	-168	42	227
352	396	352	227
42	-168	-354	-453
-388	-68	612	

sum of coefficients squared = 2288132.00

Order = 5

-102	68	98	58
-3	-54	-79	-74
-44	0	44	74
79	54	3	-58
-98	-68	102	

sum of coefficients squared = 89148.00

Order = 6

1326	-1768	-1222	234
1235	1352	729	-214
-1012	-1320	-1012	-214
729	1352	1235	234
-1222	-1768	1326	

sum of coefficients squared = 24515700.00

Order = 7

-306	646	86	-411

-425	-97	267	427
308	0	-308	-427
-267	97	425	411
-86	-646	306	

sum of coefficients squared = 2451570.00

Order = 8

34	-102	42	81
15	-57	-69	-21
42	70	42	-21
-69	-57	15	81
42	-102	34	

sum of coefficients squared = 65550.00

Order = 9

-34	136	-134	-74
85	112	7	-98
-98	0	98	98
-7	-112	-85	74
134	-136	34	

sum of coefficients squared = 164220.00

Order = 10

306	-1564	2506	-354
-1993	-266	1659	1274
-686	-1764	-686	1274
1659	-266	-1993	-354
2506	-1564	306	

sum of coefficients squared = 38779380.00

Order = 11

-204	1292	-2932	2139
1490	-1876	-1722	1127
2156	0	-2156	-1127
1722	1876	-1490	-2139
2932	-1292	204	

sum of coefficients squared = 59012100.00

Order = 12

204	-1564	4676	-6101
1140	4806	-1516	-4571
616	4620	616	-4571
-1516	4806	1140	-6101
4676	-1564	204	

sum of coefficients squared = 240432156.00

Order = 13

-153	1394	-5249	9948
-8070	-2532	8346	364
-8008	0	8008	-364
-8346	2532	8070	-9948
5249	-1394	153	

sum of coefficients squared = 667867100.00

Order = 14

51	-544	2501	-6284
8682	-4536	-4186	6604
1144	-6864	1144	6604
-4186	-4536	8682	-6284
2501	-544	51	

sum of coefficients squared = 455992020.00

Order = 15

-3	37	-203	642
-1250	1414	-546	-806
1144	0	-1144	806
546	-1414	1250	-642
203	-37	3	

sum of coefficients squared = 12546270.00

Order = 16

9	-127	817	-3144
7940	-13412	14196	-6136
-6578	12870	-6578	-6136
14196	-13412	7940	-3144
817	-127	9	

sum of coefficients squared = 1237518450.00

Order = 17

-1	16	-119	544
-1700	3808	-6188	7072
-4862	0	4862	-7072
6188	-3808	1700	-544
119	-16	1	

sum of coefficients squared = 259289580.00

Order = 18

1	-18	153	-816
3060	-8568	18564	-31824
43758	-48620	43758	-31824
18564	-8568	3060	-816
153	-18	1	

sum of coefficients squared = 9075135300.00

n = 20

Order = 1

-19	-17	-15	-13
-11	-9	-7	-5
-3	-1	1	3
5	7	9	11
13	15	17	19

sum of coefficients squared = 2660.00

Order = 2

57	39	23	9
-3	-13	-21	-27
-31	-33	-33	-31
-27	-21	-13	-3
9	23	39	57

```
                    sum of coefficients squared =      17556.00
_____
Order =  3
           -969            -357             85             377
            539             591            553             445
            287              99            -99            -287
           -445            -553           -591            -539
           -377             -85            357             969
         sum of coefficients squared =   4903140.00
_____
Order =  4
           1938            -102          -1122           -1402
          -1187            -687            -77             503
            948            1188           1188             948
            503             -77           -687           -1187
          -1402           -1122           -102            1938
         sum of coefficients squared =  22881320.00
_____
Order =  5
          -1938            1122           1802            1222
            187            -771          -1351           -1441
          -1076            -396            396            1076
           1441            1351            771            -187
          -1222           -1802          -1122            1938
         sum of coefficients squared =  31201800.00
_____
Order =  6
           1938           -2346          -1870               6
           1497            1931           1353             195
           -988           -1716          -1716            -988
            195            1353           1931            1497
              6           -1870          -2346            1938
         sum of coefficients squared =  49031400.00
_____
Order =  7
           -646            1258            306            -702
           -891            -387            321             777
            756             308           -308            -756
           -777            -321            387             891
            702            -306          -1258             646
         sum of coefficients squared =   9806280.00
_____
Order =  8
            646           -1802            510            1422
            549            -723          -1239            -735
            294            1078           1078             294
           -735           -1239           -723             549
           1422             510          -1802             646
         sum of coefficients squared =  20189400.00
_____
Order =  9
           -646            2414          -2006           -1586
            979            1993            763           -1127
          -1862            -882            882            1862
           1127            -763          -1993            -979
```

1586	2006	-2414	646

sum of coefficients squared = 47623800.00

Order = 10

646	-3094	4386	206
-3479	-1521	2219	2989
294	-2646	-2646	294
2989	2219	-1521	-3479
206	4386	-3094	646

sum of coefficients squared = 129264600.00

Order = 11

-1938	11526	-23630	12714
15453	-9407	-16821	735
16366	9702	-9702	-16366
-735	16821	9407	-15453
-12714	23630	-11526	1938

sum of coefficients squared = 3658750200.00

Order = 12

969	-6987	19091	-20897
-1826	18246	2282	-16366
-7448	12936	12936	-7448
-16366	2282	18246	-1826
-20897	19091	-6987	969

sum of coefficients squared = 3366050184.00

Order = 13

-969	8313	-28815	47807
-27086	-25266	31102	20930
-28392	-24024	24024	28392
-20930	-31102	25266	27086
-47807	28815	-8313	969

sum of coefficients squared = 14693076200.00

Order = 14

57	-573	2437	-5473
6182	-1182	-4802	2938
3848	-3432	-3432	3848
2938	-4802	-1182	6182
-5473	2437	-573	57

sum of coefficients squared = 268230600.00

Order = 15

-57	663	-3377	9663
-16182	13546	1386	-13494
6136	10296	-10296	-6136
13494	-1386	-13546	16182
-9663	3377	-663	57

sum of coefficients squared = 1756477800.00

Order = 16

19	-253	1515	-5317
11836	-16452	11564	2860
-13494	7722	7722	-13494

```
            2860              11564             -16452              11836
           -5317               1515              -253                 19
       sum of coefficients squared =     1650024600.00
```

```
Order = 17
             -19               287             -1991              8347
          -23324             44812            -57596             41548
            3094            -43758             43758             -3094
          -41548             57596            -44812             23324
           -8347              1991              -287                 19
       sum of coefficients squared =     19187428920.00
```

```
Order = 18
               1               -17               135              -663
            2244             -5508              9996            -13260
           11934             -4862             -4862             11934
          -13260              9996             -5508              2244
            -663               135               -17                  1
       sum of coefficients squared = 955277400.00
```

```
Order = 19
              -1                19              -171               969
           -3876             11628            -27132             50388
          -75582             92378            -92378             75582
          -50388             27132            -11628              3876
            -969               171               -19                  1
       sum of coefficients squared =     35345263800.00
```

APPENDIX C

DESCRIPTION OF ALGORITHM USED IN FORTRAN
PROGRAM TO GENERATE POLYNOMIALS

Acton (1970) discusses the algorithm as follows:

Without delving into the proof, for which see Ralston (1965), we say immediately that polynomials $p(x_i)$ orthogonal with respect to summation over the points $\{x_i\}$ are easily generated from a three-term recurrence relation. More remarkable, the points need not be uniformly or even regularly spaced. **Any set of arbitrary points has its corresponding set of orthogonal polynomials.** The points may even be used with arbitrary weights, though we shall ignore this embellishment, restricting ourselves to uniform weighting. Again, Ralston gives the more general treatment.

The fundamental recurrence relation is

$$p_{j+1}(x) = (x-a_j) \cdot p_j(x) - b_j \cdot p_{j-1}(x) \tag{1}$$

where $\qquad p_0(x) = 1$

and $\qquad p_1(x) = (x - a_0) \quad$ (or $b_0 = 0$).

These relations, being identities in $\underline{x}$, hold for all $\underline{x}$, not merely for the $\underline{m}$ points $\{x\}$.

The orthogonality relations have the form

summation from $i=1$ to m of $p_k(x_i) \cdot p_j(x_i) = 0$, k not$=j$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad = g_k$, [when] $k = j$

and these, applied to [equation (1) above], give

$$g_j = \text{summation } i=1 \text{ to } m \text{ of } [p_j(x_i)]^2 \qquad j=0,1,\ldots$$

$$a_j = \frac{\text{summation } i=1 \text{ to } m \text{ of } x_i[p_j(x_i)]^2}{\text{summation } i=1 \text{ to } m \text{ of } [p_j(x_i)]^2}$$

$$= \text{summation } i=1 \text{ to } m \text{ of } \frac{x_i p^2_j(x_i)}{g_j}$$

$$b_j = \frac{\text{summation } i=1 \text{ to } m \text{ of } [p_j(x_i)]^2}{\text{summation } i=1 \text{ to } m \text{ of } [p_{j-1}(x_i)]^2}$$

$$= \frac{g_j}{g_{j-1}} \qquad j=1,2,\ldots$$

as the computational formulae. The calculations may be arranged in a convenient array

	x_1	x_2	x_3	$\ldots$	x_m	a	b	g	x
$p_0(x_i)$	1	1	1		1	a_0	0	m	1
$p_1(x_i)$	(x_1-a_0)	$\ldots$				a_1	b_1	g_1	
$p_2(x_i)$						a_2	b_2	g_2	
.						.	.	.	
.						.	.	.	
.						.	.	.	
$p_{m-1}(x_i)$									
	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\ldots$	$f(x_m)$				

We may calculate g_0, a_0, $p_1(x_i)$, g_1, b_1, a_1, $p_2(x_i)$, and so on, in order -- using [equation (1)] for the several $p_j(x_i)$. Once the a_j and b_j are available, the $p_j(x)$ for any arbitrary given $\underline{x}$ may also be generated by the recurrence [equation (1)].

Acton, pp. 236-7.

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